Parafermionic and Generalized Parafermionic Algebras

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Abstract: The general properties of the ordinary and generalized parafermionic algebras are discussed. The generalized parafermionic algebras are proved to be polynomial algebras. The ordinary parafermionic algebras are shown to be connected to the Arik–Coon oscillator algebras.

The study of systems of many spins is of interest in many branches of physics. This study is in many cases facilitated through boson mapping procedures (see [1] for a comprehensive review). Some well-known examples are the Holstein–Primakoff mapping of the spinor algebra onto the harmonic oscillator algebra [2] and the Schwinger mapping of Lie algebras (or of q-deformed algebras) onto the usual (or onto the q-deformed) oscillator algebras [3].

In parallel, in addition to bosons and fermions, parafermions of order p have been introduced [4, 5] (with p being a positive integer), having the characteristic property that at most p identical particles of this kind can be found in the same state. Ordinary fermions clearly correspond to parafermions with

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$p = 1$, since only one fermion can occupy each state according to the Pauli principle. While fermions obey Fermi–Dirac statistics and bosons obey Bose–Einstein statistics [6], parafermions are assumed to obey an intermediate kind of statistics, called parastatistics [6, 7, 8].The notion of parafermionic algebras has been recently enlarged by Quesne [9], while the relation between parafermionic algebras and other algebras has been given in [10, 11, 12]. The properties of parafermions and parabosons, as well as the parastatistics and field theories associated with them, have been the subject of many recent investigations [13, 14]. Parafermions and parabosons have also been involved in mapping studies. A mapping of the spinor algebra onto a parafermionic algebra has been discussed in [4, 5, 7]. Mappings of so(2n), sp(2n,R), and other Lie algebras onto parafermionic and parabosonic algebras have been studied in [7, 15], while parabosonic mappings of osp(m,n) superalgebras have been given in [16].

Recently [17] the algebras of the operators of a single spinor with fixed spin value j have been mapped onto polynomial algebras, which constitute a quite recent subject of investigations in physics [9, 12, 18]. In polynomial algebras the commutator of two generators does not result in a linear combination of generators, as in the case of the usual Lie algebras, but rather into a combination of polynomials of the generators. The mappings of ref. [17] connect the class of spinor algebras to the class of polynomial algebras.

In the present work we show that the polynomial algebras of ref. [17], which are connected to the single spinor algebras, are indeed examples of either parafermionic algebras [4, 5, 7] or generalized parafermionic algebras [9, 12].

Let us start by defining the algebra $\mathcal{A}_n^{[p]}$, corresponding to n parafermions of order p. This algebra is generated by n parafermionic generators $b_i, b_i^\dagger$, where $i = 1, 2, \ldots, n$, satisfying the *trilinear* commutation relations:

$$\left[M_{k\ell}, b_m^\dagger\right] = \delta_{\ell m} b_k^\dagger, \quad [M_{\ell k}, b_m] = -\delta_{\ell m} b_k, \tag{1}$$

where $M_{k\ell}$ is an operator defined by:

$$M_{k\ell} = \frac{1}{2}\left(\left[b_k^\dagger, b_\ell\right] + p\delta_{k\ell}\right). \tag{2}$$

From this definition it is clear that eq. (1) is a *trilinear* relation, i.e. a relation relating three of the operators $b_i^\dagger, b_i$. Finally the definition of the parafermionic algebra is completed by the relation:

$$[b_i, [b_j, b_k]] = \left[b_i^\dagger, \left[b_j^\dagger, b_k^\dagger\right]\right] = 0. \tag{3}$$

Each parafermion separately is characterized by the ladder operators $b_i^\dagger$ and b_i and the number operator M_{ii}. The basic assumption is that the parafermionic creation and annihilation operators are nilpotent ones:

$$(b)^{p+1} = \left(b^\dagger\right)^{p+1} = 0. \tag{4}$$

In ref. [10] it is proved that the single parafermionic algebra is a generalized oscillator algebra [19], satisfying the following relations (for simplicity we omit the parafermion indices):

$$\left[M, b^\dagger\right] = b^\dagger, \quad [M, b] = -b, \tag{5}$$

$$b^\dagger b = [M] = M\left(p+1-M\right), \quad bb^\dagger = [M+1] = (M+1)\left(p-M\right), \tag{6}$$

$$M\left(M-1\right)\left(M-2\right)...\left(M-p\right) = 0. \tag{7}$$

The definition (2) – or equivalently eq. (6) – implies the commutation relation:

$$\left[b^\dagger, b\right] = 2(M - p/2). \tag{8}$$

The above relation combined with (5) suggests the use of the parafermions as spinors of spin $p/2$

$$S_+ \leftrightarrow b^\dagger, \quad S_- \leftrightarrow b, \quad S_o \leftrightarrow (M - p/2). \tag{9}$$

The Cayley identity is also valid:

$$\prod_{k=-p/2}^{p/2} (S_o - k) = 0.$$

It is worth noticing that in the case of parafermions the commutation relation (8) is some how trivial because it is inherent in the definition of the number operator (2). This relation switches the trilinear commutation relations to ordinary commutation relations, where two operators are involved.

In contrast, in the case of parabosons this construction is not trivial, because anticommutation relations are involved in the definition of the number operator[14].

We start now examining in detail the connection between spinors with $j = p/2$ and parafermions of order p.

The $p = 1$ parafermions coincide with the ordinary fermions, i.e. the usual spin 1/2 spinors [20].

For spinors with $j = 1$ Chaichian and Demichev [17] use the following mapping

$$S_+ \leftrightarrow \sqrt{2}a^\dagger, \qquad S_- \leftrightarrow \sqrt{2}a, \tag{10}$$

where

$$a^3 = {a^\dagger}^3 = 0, \tag{11}$$

$$aa^\dagger + {a^\dagger}^2 a^2 = 1. \tag{12}$$

Using the above two relations we can define the number operator N

$$N = 1 - \left[a, a^\dagger\right] = a^\dagger a + {a^\dagger}^2 a^2. \tag{13}$$

This number operator satisfies the linear commutation relations:

$$\left[N, a^\dagger\right] = a^\dagger, \quad [N, a] = -a.$$

The self-contained commutation relations for the $p = 2$ parafermions are given in ref. [7] (eqs (5.13) to (5.20))

$$b^3 = {b^\dagger}^3 = 0, \tag{14}$$

$$bb^\dagger b = 2b, \tag{15}$$

$$b^\dagger b^2 + b^2 b^\dagger = 2b. \tag{16}$$

The set of relations (14)–(16) imply the following definition of the number operator M:

$$M = \frac{1}{2}\left(\left[b^\dagger, b\right] + 2\right). \tag{17}$$

The set of relations (11)–(12) imply relations (14)–(16) after taking into consideration the correspondence:

$$a = \frac{1}{\sqrt{2}}b, \qquad a^\dagger = \frac{1}{\sqrt{2}}b^\dagger. \tag{18}$$

For example one can easily see the following:

a) Eq. (14) occurs trivially from eq. (11).

b) Eq. (15) is obtained by multiplying eq. (12) by a on the right and using eq. (11).

c) Eq. (16) is obtained by multiplying eq. (12) by a on the left and using eq. (11) and (12).

In ref. [10] the parafermionic algebra (14)–(17) was shown to be equivalent to the deformed oscillator algebra [19], which is defined by relations (4)–(7), for $p = 2$. This deformed oscillator algebra satisfies in addition the relations (11) to (13). Therefore the Chaichian - Demichev polynomial algebra (11)–(13), the $p = 2$ parafermionic algebra (14)–(17) and the deformed oscillator algebra (4)–(7) are equivalent.

Relations (12) and (13) indicate that $aa^\dagger$ and N can be expressed as a linear combination of monomials $\left(a^\dagger\right)^k a^k$. This is the reason the algebra described by eqs (12)–(13) is called in [17] a "polynomial" algebra.

What we have just seen is that the polynomial algebra (11)–(13) is in fact the $p = 2$ parafermionic algebra (14)–(17). The new result which arises from this discussion is that the parafermionic algebra can be written as a polynomial algebra through the r.h.s of eq. (13). It seems that this fact has been ignored, while the "dual" relation, giving $b^\dagger b$ or $bb^\dagger$ as polynomial functions of the number operator,

$$b^\dagger b = M(3 - M), \qquad bb^\dagger = (M + 1)(2 - M),$$

is known [9, 10, 11].

For spinors with $j = 3/2$ Chaichian and Demichev [17] use the following mapping

$$S_+ \leftrightarrow \sqrt{3}a^+, \quad S_- \leftrightarrow \sqrt{3}a, \tag{19}$$

where

$$a^4 = a^{\dagger 4} = 0, \tag{20}$$

$$aa^\dagger = 1 + \frac{1}{3}a^\dagger a - \frac{1}{3}a^{\dagger 2}a^2 - \frac{2}{3}a^{\dagger 3}a^3, \tag{21}$$

$$\left[a, a^\dagger\right] = 1 - \frac{2}{3}N. \tag{22}$$

$$\tag{23}$$

The last two equations imply the following expansion of the number operator:

$$N = a^\dagger a + \frac{1}{2} {a^\dagger}^2 a^2 + {a^\dagger}^3 a^3. \tag{24}$$

These relations are the analogues of eqs. (11)–(13) for the $j = 3/2$ case.

The complicated self-consistent commutation relations for the $p = 3$ parafermionic algebra are given in Appendix B of ref. [7]. After long but straightforward calculations the $p = 3$ parafermionic relations are deduced from the above eqs (21)–(24) by taking into account the correspondence:

$$a = \frac{1}{\sqrt{3}} b, \quad a^\dagger = \frac{1}{\sqrt{3}} b^\dagger. \tag{25}$$

Therefore the polynomial algebra (21)–(24) is in fact the $p = 3$ parafermionic algebra. The new result which again arises from this discussion is that the parafermionic algebra can be written as a polynomial algebra through eq. (24), while the "dual" relation

$$b^\dagger b = M(4 - M), \quad bb^\dagger = (M + 1)(3 - M),$$

is again already known [10].

Stimulated by the above results we can show the following proposition:

Proposition 1 *The $j = p/2$ spinor algebra $\{S_\pm, S_o\}$ is mapped onto the p-parafermionic algebra $\{b^+, b, M\}$ which is a polynomial algebra given by the relations:*

$$\begin{array}{c} \left[M, b^\dagger\right] = b^\dagger, \\ \left[M, b\right] = -b, \\ b^{p+1} = \left(b^\dagger\right)^{p+1} = 0, \\ b^\dagger b = M\left(p + 1 - M\right) = \left[M\right], \\ bb^\dagger = \left(M + 1\right)\left(p - M\right) = \left[M + 1\right], \\ M = \frac{1}{2}\left(\left[b^\dagger, b\right] + p\right), \end{array} \tag{26}$$

where the number operator M is given by the following polynomial relation

$$M = \sum_{k=1}^{p} \frac{c_k}{p^k} {b^\dagger}^k b^k. \tag{27}$$

With the "factorial" $[k]!$ *being defined as*

$$[0]! = 1, \quad [n]! = [n]\,[n-1]! = \prod_{\ell=1}^{n} [\ell] = \frac{n!p!}{(p-n)!},$$

the coefficients c_1, c_2, $\ldots, c_p$ *can be determined from the solution of the system of equations:*

$$\left.\begin{array}{l} \rho(1) = 1 \\ \rho(2) = 2 \\ \ldots \\ \rho(p) = p \end{array}\right\} \tag{28}$$

where

$$\rho(n) = \sum_{k=1}^{n} \frac{c_k}{p^k} \frac{\Gamma(n+1)}{\Gamma(n-k+1)} \frac{\Gamma(p+k-n+1)}{\Gamma(p-n+1)}.$$

This is true because we can see that

$$b^{\dagger k} b^k = \prod_{\ell=0}^{k-1} [M-\ell] \equiv \frac{\Gamma(M+1)}{\Gamma(M-k+1)} \frac{\Gamma(p+k-M+1)}{\Gamma(p-M+1)}.$$

The fact that the number operator of a parafermionic algebra can be written as a combination of monomials, i.e. eq. (27), was not previously known in the context of parafermionic algebras. The polynomial expressions as in eq. (27) are similar to the ones used for the construction of the projection operators in the case of the su(2) algebra [21, 22]. The projection operator method has also been used in the case of the $\mathrm{su}_q(2)$ and $\mathrm{su}_q(1,1)$ algebras [23] as a dynamic tool for the calculation of the Clebsch-Gordan coefficients. On the other hand the parafermionic algebra is a finite dimensional realization of the su(2) algebra, coinciding with the spinor algebra.

The analytic calculation of the coefficients c_k can be achieved by expanding the number operator M in a sum over the projection operators

$$P_m \left| n \right\rangle = \delta_{nm} \left| n \right\rangle$$

in the following way:

$$M = \sum_{m=1}^{p} m P_m$$

The projection operator P_o to the lowest weight eigenvalue is given by the expression:

$$P_o = \sum_{k=0}^{p} d_k \left(b^+\right)^k b^k,$$

while

$$P_m = \frac{1}{[m]!} \left(b^+\right)^m P_o b^m = \frac{1}{[m]!} \sum_{k=0}^{p-m} d_k \left(b^+\right)^k b^k,$$

where the coefficients d_n are given by the recurrence formula

$$\begin{array}{l} d_0 = 1, \\ d_n = -\sum\limits_{k=0}^{n-1} \frac{d_k}{[n-k]!}. \end{array}$$

Then the general solution is given by:

$$d_n = \sum_{i=1}^{n} (-1)^i \left(\sum_{\substack{0 < k_1, k_2, \ldots, k_i \leq n \\ k_1 + k_2 + \ldots + k_i = n}} \frac{1}{[k_1]!\,[k_2]! \ldots [k_i]!} \right) \tag{29}$$

We must point out that these formulae are not specific to the chosen parafermion structure function $[x] = x(p+1-x)$ and can be applied for any parafermionic oscillator stucture function.

The number operator M can be expressed using the projection operators:

$$M = \sum_{m=1}^{p} m P_m = \sum_{k=1}^{p} \frac{c_k}{p^k} \left(b^+\right)^k b^k,$$

while the coefficients c_n can be fould to be

$$c_n = p^n \sum_{k=1}^{n} \frac{k}{[k]!} d_{n-k}.$$

In Table 1 the coeffients up to $p = 5$ are explicitly given.

One must notice that the parafermionic algebra (5–7) has affinities with the Arik – Coon Q-deformed algebra [27], which is defined by the relations:

$$[N, a] = -a, \quad [N, a^\dagger] = a^\dagger, \quad a^\dagger a = [N]_Q, \quad a a^\dagger = [N+1]_Q \tag{30}$$

Table 1: Coefficients appearing in eq. (27).

p	c_1	c_2	c_3	c_4	c_5
1	1	$--$	$--$	$--$	$--$
2	1	1	$--$	$--$	$--$
3	1	1/2	1	$--$	$--$
4	1	1/3	1/3	7/9	$--$
5	1	1/4	1/6	19/96	23/48

where $[x]_Q = (1-Q^x)/(1-Q)$. The generators of this oscillator satisfy the following commutation relation

$$[a, a^\dagger] = Q^N \tag{31}$$

By defining $Q = \exp[-\tau]$ the commutation relation (31) can be written, for $\tau \longrightarrow 0$ as

$$[a, a^\dagger] = \exp[-\tau N] = 1 - \tau N + \mathcal{O}(\tau^2) \tag{32}$$

After comparing the above equation with equation (8), we can see that there is a approximative mapping of the parafermionic oscillator to the Arik – Coon oscillator by putting

Arik – Coon	$\longrightarrow$	para Fermi
b	$\longrightarrow$	$\sqrt{p}\ a$
$b^\dagger$	$\longrightarrow$	$\sqrt{p}\ a^\dagger$
M	$\longrightarrow$	N
τ	$=$	$2/p$

The meaning of the order p of the parafermionic oscillator is quite clear, p is a the "*capacity*" of the oscillator, i.e. the maximum number of permitted states, which can be found similtaneously at the same position. Therefore, the parameter Q of the Arik – Coon oscillator is a "*measure* " of the similtaneously existed states at the same position.

A nice example is the case of the $J = 0$ pairing of nucleons in a closed nuclear shell. The algebra of the fermion pairs coupled to angular momentum zero are descibed by the algebra [1]:

$$\left[A_0, A_0^+\right] = 1 - N_F/\Omega, \quad \left[\frac{N_F}{2}, A_0^+\right] = A_0^+, \quad \left[\frac{N_F}{2}, A_0\right] = -A_0$$

where N_F is the number of fermions, $2\Omega = 2j + 1$ is the size of the shell, i.e. the "capacity" of our space. The simplest piring Hamiltonian is given by:

$$H = -G\Omega A_0^+ A_0$$

For the above algebra there is a natural mapping to parafermions of order p, each parafermion corresponds to a Fermi pair [28] and $p = \Omega$. The ordinary q-deformed oscillator [3] fails to give an approximation of the pairing model, while the Arik – Coon oscillator is quite satisfactory [29].

Conclusions The parafermionic algebras can be considered as polynomial algebras, their diagonal number operator M_{ii} being able to be written as a combination of monomials of the ladder operators. The general problem of finding an expression of the number operator M_{ij} as a combination of monomials of the ladder operators is still open. A similar problem exists in quonic algebras [24, 25, 26]. Work in this direction is in progress. The Arik – Coon deformed oscillator is a fair approximation of the parafermionic algebra.

Support from the Greek Secretariat of Research and Technology under contract PENED 95/1981 is gratefully acknowledged.

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